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# ON REDUCING THE CONTINUATION PROBLEM FOR AN ELLIPTIC EQUATION TO AN OPTIMAL CONTROL PROBLEM AND ITS STUDY

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**Abstract.** In the paper we consider a continuation problem for a second order elliptic equation. This problem is treated as an inverse one to some direct problem. In the direct problem it is required to determine the solution of the equation according to the known function given on a part of the boundary of the considered domain. The inverse problem is to determine the unknown function according to the additional information. This problem is reduced to the optimal control problem. In the obtained problem a theorem on the existence optimal control, Freshet differentiability of the functional, necessary and sufficient optimality condition in the form of variation inequality is proved.

**Keywords**: Ill-posed problem, continuation problem, optimal control problem, optimality condition.
**AMS Subject Classification:**
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## 1 Introduction

As it is known, the Cauchy problem for the Laplace equation is an ill-posed problem (Vladimirov, 1971). Nevertheless this problem turned out so important to practice that lot of works were devoted to its study (Lattes & Lions, 1970) and it has a wide practical application, for example, in the problem of determining the electrostatic field inside the Earth there arises a two-dimensional Cauchy problem for the Laplace equation. Therefore various research methods are used to study this problem and similar problems for second order elliptic equations. In most cases, these problems are treated as inverse ones to direct and well-posed problems. Further unknown functions are restored according to additional information on the solution of direct problems. There are various methods for studying such problems. One of these methods is the construction of discrepancy functional according to the additional information and study of the problem of minimization of this functional under natural restrictions. There exist continuation problems for elliptic equations (Lattes & Lions, 1970; Kabanikhin, 2009) that are also ill-posed, and special case of which is the above mentioned Cauchy problem for the Laplace equation. Note that close problems for elliptic and hyperbolic equations were considered in Tagiyev & Kasumova (2017); Lapin & Hasanov (2010); Kuliev & Nasibzade (2018); Tagiev & Kasymova (2018); Romanov & Shamaev (2020); Shamaev & Romanov (2021); Kuliev & Askerov (2022).

In the present paper we consider a continuation problem for a second order elliptic equation in multi-dimensional case. This problem is treated as inverse one to the direct problem. In the direct problem it is required to determine the solution of the equation according to the known function given on a part of the boundary of the considered domain. The inverse problem is to determine the unknown function according to the additional information. This problem is reduced to an optimal control problem and is studied by the methods of theory of optimal control.

## 2 Problem statement

In the cylinder $\Omega = \left\{(x, y) \in R^{n+1} : x \in (0, l)\,, y \in D \subset R^n\right\}$ we consider the initial-boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}\left(y\right)\frac{\partial u}{\partial y_j}\right) - a\left(y\right)u = f\left(x, y\right), \ \ (x, y) \in \Omega, \tag{1}$$

$$u(0, y) = \varphi(y), \frac{\partial u(0, y)}{\partial x} = 0, y \in D, \tag{2}$$

$$\left.\frac{\partial u}{\partial \nu_A}\right|_{\partial D} = 0, x \in (0, l)\,. \tag{3}$$

Here $D \subset R^n$ is a bounded domain with enough smooth boundary $\partial D, l > 0$ is a given number, $f \in L_2\left(\Omega\right), \varphi \in L_2\left(D\right)$ are the given functions, the coefficients $a_{ij}\left(y\right), i, j = \overline{1, n}, a\left(y\right)$ are the given functions and they possess the following properties:

$$a_{ij} \in C^1\left(\overline{D}\right), a \in C\left(\overline{D}\right), \ \ a_{ij}\left(y\right) = a_{ji}\left(y\right), \ \ i, j = \overline{1, n}, a\left(y\right) \geq 0, y \in \overline{D}$$

and

$$\sum_{i,j=1}^{n} a_{ij}\left(y\right)\xi_i\xi_j \geq \alpha \sum_{k=1}^{n} \xi_k^2$$

for any $\xi \in R^n$ and for all $y \in \overline{D}$, $\alpha = const > 0$;

$$\frac{\partial u}{\partial \nu_A} \equiv \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial y_j}\cos\left(\nu, y_i\right)$$

is a conormal derivative, $\nu$ is a unit normal to $\partial D$.

Problem (1)-(3) is a continuation problem (Lattes & Lions, 1970; Kabanikhin, 2009) and it is an ill-posed problem.

**Example.** Let us consider a special case of problem (1)-(3), i.e. in the quadrate

$$\Omega = \left\{(x, y) \in R^2 : x \in (0, 1)\,, y \in (0, 1)\right\}$$

we consider the initial-boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ \ (x, y) \in \Omega, \tag{4}$$

$$u(0, y) = \varphi(y), \frac{\partial u(0, y)}{\partial x} = 0, y \in (0, 1), \tag{5}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0, \ \ x \in (0, 1)\,. \tag{6}$$

The problem under consideration is Adamard ill-posed. Show that it is unstable. Indeed, for

$$\varphi(y) = \frac{1}{k}\cos\pi k y$$

the solution of problem (4)-(6) is expressed by the formula

$$u(x,y) = \frac{1}{k}\cos\pi ky ch\pi kx.$$

Consequently, with increasing $k$ the function $\varphi(y)$ can be arbitrary small, while the solution $u(x,y)$ unboundedly increases as $k\to\infty$.

Let us consider the ill-posed problem (1)-(3) as inverse to the following direct problem

$$\frac{\partial^2 u}{\partial x^2} + \sum_{i,j=1}^{n}\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u}{\partial y_j}\right) - a(y)\,u = f(x,y),\ \ (x,y)\in\Omega, \tag{7}$$

$$\frac{\partial u(0,y)}{\partial x} = 0, \frac{\partial u(l,y)}{\partial x} = v(y), y\in D, \tag{8}$$

$$\left.\frac{\partial u}{\partial \nu_A}\right|_{\partial D} = 0, x\in(0,l). \tag{9}$$

In the direct problem (7)-(9) it is required to determine the function $u(x,y)$ in $\Omega$ according to the known function $v(y)\in L_2(D)$, given on a part of the boundary $x=l$ of the considered domain $\Omega$. The inverse problem is to determine the functions $v(y)$ from the relations (7)-(9) according to the additional information

$$u(0,y) = \varphi(y), y\in D. \tag{10}$$

We will consider the generalized solution of problem (7)-(9).

For the given function $v(y)\in L_2(D)$ under the generalized solution of problem (7)-(9) we understand the function $u=u(x,y)=u(x,y;v)\in W_2^1(\Omega)$, $\omega\in W_2^1(\Omega)$ that for any $\omega\in W_2^1(\Omega)$ satisfies the following integral identity $\omega\in W_2^1(\Omega)$

$$\int\limits_{\Omega}\left[-\frac{\partial u}{\partial x}\cdot\frac{\partial \omega}{\partial x} - \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial y_j}\cdot\frac{\partial\omega}{\partial y_i} - au\omega\right]dxdy+$$

$$+\int\limits_{D} v(y)\omega(l,y)dy = \int\limits_{\Omega} f\omega dxdy. \tag{11}$$

## 3 Some auxiliary facts

Acting as in Mikhailov (1983), p. 208, under the given conditions of the data of problem and (7)-(9) we can prove the following theorem.

**Theorem 1.** *There exists a unique generalized solution of the problem (7)-(9) $u(x,y)\in W_2^1(\Omega)$ and the following estimation is valid*

$$\|u\|_{W_2^1(\Omega)} \le c\left(\|f\|_{L_2(\Omega)} + \|\nu\|_{L_2(D)}\right), \tag{12}$$

*here and in that follows, by $c$ we will denote various constants independent of the estimated quantities.*

Further, theorem 1 Mikhailov (1983), p.149 yields.

**Theorem 2.** *The solution of the problem (7)-(9) $u(x,y)$ has the trace $u(0,y)$, belonging to $L_2(D)$, and we have the following inequality*

$$\|u(0,y)\|_{L_2(D)} \le c\,\|u\|_{W_2^1(\Omega)}. \tag{13}$$

*Then it follows from inequalities (12) and (13) that*

$$\|u(0,y)\|_{L_2(D)} \le c\left[\|f\|_{L_2(\Omega)} + \|v\|_{L_2(D)}\right]. \tag{14}$$

# 4 Reducing the inverse problem to an optimal control problem

Now we reduce the inverse problem of finding the function $v(y)$ to the following optimal control problem: to find such a function $v(y)$ from the class

$$V = \{v(y) : v \in L_2(D), a \leq v(y) \leq b \text{ for almost everywhere on } D\},$$

that together with the solution of problem (7)-(9) it give minimum to the functional

$$J_0(v) = \frac{1}{2}\int_D [u(0, y; v) - \varphi(y)]^2 \, dy, \tag{15}$$

where $u(x, y; v)$ is the solution of problem (7)-(9) for $v = v(y), a, b$ are a given numbers, $a < b$.

We call the function $v(y)$ a control, $V$ a class of admissible controls.

We call this problem the problem (7)-(9),(15). Between the problems (7)-(10) and (7)-(9),(15) there is a close connection, if in the problem (7)-(9),(15) $\min\limits_{v\in V} J_0(v) = 0$, then the additional condition (10) is fulfilled.

**Theorem 3.** *Let the conditions accepted in the statement of problem (7)-(9),(15) be fulfilled. Then the set of optimal controls problem (7)-(9),(15)*

$$V_* = \{v_* \in V : J_0(v_*) = J_{0^*} = \inf\{J_0(v) : v \in V\}\}$$

*is not empty, weakly compact in $L_2(D)$ and any minimizing sequence $\{v_m\} \subset V$ weakly in $L_2(D)$ converges to the set $V_*$.*

*Proof.* The set $V$ is convex, closed and bounded in $L_2(D)$, so weakly compact in $L_2(D)$. Show that the functional (15) weak in $L_2(D)$is continuous on the set $V$.

Let $v \in V$ be some element and $\{v_m\} \subset V$ be an arbitrary sequence

$$\text{such that } v_m \to v \text{ weakly in } L_2(D) \text{ as } m \to \infty. \tag{16}$$

By the unique solvability of boundary value problem (7)-(9) to each control $v_m \in V$ there corresponds a unique solution $u_m = u(x, y; v_m)$ of the problem (7)-(9) and the following estimation is valid (see. (12))

$$\|u_m\|_{W_2^1(\Omega)} \leq c\left(\|v_m\|_{L_2(D)} + \|f\|_{L_2(\Omega)}\right) \leq c.$$

Then from the imbedding theorem [7, p.64] it follows that from the sequence $\{u_m\}$ one can choose such a subsequence $\{u_{m_k}\}$ that as $k \to \infty$

$$u_{m_k} \to u \text{ strongly in } L_2(\Omega), \tag{17}$$

$$\frac{\partial u_{m_k}}{\partial x} \to \frac{\partial u}{\partial x}, \quad \frac{\partial u_{m_k}}{\partial y} \to \frac{\partial u}{\partial y} \text{ weakly in } L_2(\Omega), \tag{18}$$

$$u_{m_k}(0, y) \to u(0, y) \text{ strongly in } L_2(D), \tag{19}$$

where $u = u(x, y) \in W_2^1(\Omega)$ is some element.

Show that $u(x, y) = u(x, y; v)$, i.e. the function $u(x, y)$ is the solution of problem (7)-(9), corresponding to the control $v \in V$. It is clear that the following identity

$$\int_\Omega \left[-\frac{\partial u_{m_k}}{\partial x} \cdot \frac{\partial \eta}{\partial x} - \sum_{i,j=1}^{n} a_{ij}\frac{\partial u_{m_k}}{\partial y_j} \cdot \frac{\partial \eta}{\partial y_i} - a u_{m_k}\eta\right] dxdy+$$

$$+\int_D v_{m_k}(y)\,\eta(l,y)\,dy=\int_\Omega f\eta dxdy, \tag{20}$$

is valid for all $\eta\in W_2^1(\Omega)$.

Passing to limit in (20) as $k\to\infty$ and using (16)-(18) we get that the function $u(x,y)$ satisfies the identity (11). Hence and from the uniqueness of the solution of problem (7)-(9), corresponding to the control $v\in V$ it follows that $u(x,y)=u(x,y;v)$. Using the uniqueness of the solution of problem (7)-(9), corresponding to the control $v\in V$, it is easy to verify that relations (17), (18), (19) are valid not only for the subsequence $\{u_{m_k}\}$, but also for all the sequence $\{u_m\}$. Consequently, in particular, the limit relation $u_m(0,y)\to u(0,y)$ strongly in $L_2(D)$. Using this relation, from (15) we get that $J_0(v_m)\to J_0(v)$ as $m\to\infty$, i.e. $J_0(v)$ weakly in $L_2(D)$ continuous on the set $V$. Then by theorems 2 and 4 from [8, p.49, p. 51], we get that all the statements of theorem 3 one valid. Theorem 3 is proved. □

In the future in order to avoid the degeneration of the obtained necessary and sufficient condition of optimality, we regularize the functional (15):

$$J_\beta(v)=J_0(v)+\frac{\beta}{2}\int_D|v(y)|^2\,dy, \tag{21}$$

where $\beta>0$ is a given number. Now it is required to find the minimum of the functional (21) in the class $V$ under the constraints (7)-(9). Since the problem (7)-(9) is linear, the functional (21) is quadratic and strong in $L_2(D)$, in the problem (7)-(9), (21) there exists a unique optimal control in the class $V$ [9, p.54].

## 5 Differentiability of the functional (21)

Let $v\in V$ and $v+\delta v\in V$ be two admissible controls, corresponding solutions of problem (7)-(9) denote by $u(x,y;v)$ and $u(x,y;v+\delta v)$. Let $\delta u(x,y)=u(x,y;v+\delta v)-u(x,y;v)$.

It is clear that the function $\delta u(x,y)$ is the solution of the following problem

$$\frac{\partial^2\delta u}{\partial x^2}+\sum_{i,j=1}^n\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial\delta u}{\partial y_j}\right)-a(y)\,\delta u=0,\ \ (x,y)\in\Omega, \tag{22}$$

$$\frac{\partial\delta u(0,y)}{\partial x}=0,\ \frac{\partial\delta u(l,y)}{\partial x}=\delta v(y), y\in D, \tag{23}$$

$$\left.\frac{\partial\delta u}{\partial\nu_A}\right|_{\partial D}=0, x\in(0,l). \tag{24}$$

For this problem we get the following analogues of inequalities (12), (13), (14)

$$\|\delta u\|_{W_2^1(\Omega)}\le c\,\|\delta v\|_{L_2(D)},\quad \|\delta u(0,y)\|_{L_2(D)}\le c\,\|\delta u\|_{W_2^1(\Omega)},\quad \|\delta u(0,y)\|_{L_2(D)}\le c\,\|\delta v\|_{L_2(D)}. \tag{25}$$

We calculate the increment of the functional (21).

It is clear that

$$\Delta J_\beta(v)=J_\beta(v+\delta v)-J_\beta(v)=$$

$$=\frac{1}{2}\int_D\left\{[u(0,y;v+\delta v)-\varphi(y)]^2-[u(0,y;v)-\varphi(y)]^2\right\}dy+\frac{\beta}{2}\int_D\left[(v+\delta v)^2-v^2\right]dy=$$

$$=\int_D[u(0,y;v)-\varphi(y)]\,\delta u(0,y)\,dy+\beta\int_D v(y)\,\delta v(y)\,dy+R, \tag{26}$$

where
$$R=\frac{1}{2}\int\limits_{D}|\delta u\left(0,y\right)|^{2}dy+\frac{\beta}{2}\int\limits_{D}|\delta v\left(y\right)|^{2}dy \tag{27}$$

is a remainder term.

We introduce the adjoint problem to the problem (7)-(9),(21)

$$\frac{\partial^2\psi}{\partial x^2}+\sum_{i,j=1}^{n}\frac{\partial}{\partial y_i}\left(a_{ij}\left(y\right)\frac{\partial\psi}{\partial y_j}\right)-a\left(y\right)\psi=0,\ \ (x,y)\in\Omega, \tag{28}$$

$$\frac{\partial\psi(0,y)}{\partial x}=-\left[u\left(0,y;v\right)-\varphi\left(y\right)\right],\frac{\partial\psi(l,y)}{\partial x}=0,y\in D, \tag{29}$$

$$\left.\frac{\partial\psi}{\partial\nu_A}\right|_{\partial D}=0,x\in\left(0,l\right). \tag{30}$$

Let us consider the generalized solution of problem (28)-(30). Under the generalized solution of problem (28)- (30) we understand the function $\psi=\psi\left(x,y\right)=\psi\left(x,y;v\right)\in W_2^1\left(\Omega\right)$, that for any function $g\left(x,y\right)\in W_2^1\left(\Omega\right)$satisfies the integral identity

$$\int\limits_{\Omega}\left[-\frac{\partial\psi}{\partial x}\cdot\frac{\partial g}{\partial x}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial\psi}{\partial y_j}\cdot\frac{\partial g}{\partial y_i}-a\left(y\right)\psi g\right]dxdy+$$

$$+\int\limits_{D}\left[u\left(0,y;v\right)-\varphi\left(y\right)\right]g\left(0,y\right)dy=0. \tag{31}$$

Under the solution of problem (22)-(24) we understand the function $\delta u\left(x,y\right)\in W_2^1\left(\Omega\right)$, that for any $\omega\left(x,y\right)\in W_2^1\left(\Omega\right)$ satisfied the integral identity

$$\int\limits_{\Omega}\left[\frac{\partial\delta u}{\partial x}\cdot\frac{\partial\omega}{\partial x}+\sum_{i,j=1}^{n}a_{ij}\frac{\partial\delta u}{\partial y_j}\cdot\frac{\partial\omega}{\partial y_i}+a\left(y\right)\delta u\omega\right]dxdy-\int\limits_{D}\delta v\left(y\right)\omega\left(l,y\right)dy=0. \tag{32}$$

If in the identity (31) we assume $g\left(x,y\right)=\delta u\left(x,y\right)$, and in the identity (32) put $\omega\left(x,y\right)=\psi\left(x,y\right)$ and the obtained relations, we have

$$\int\limits_{D}\left[u\left(0,y;v\right)-\varphi\left(y\right)\right]\delta u\left(0,y\right)dy-\int\limits_{D}\delta v\left(y\right)\psi\left(l,y\right)dy=0$$

or
$$\int\limits_{D}\left[u\left(0,y;v\right)-\varphi\left(y\right)\right]\delta u\left(0,y\right)dy=\int\limits_{D}\delta v\left(y\right)\psi\left(l,y\right)dy. \tag{33}$$

If we take into account formula (33) in (26), we get

$$\Delta J_\beta\left(v\right)=\int\limits_{D}\psi\left(l,y\right)\delta v\left(y\right)dy+\beta\int\limits_{D}v\left(y\right)\delta v\left(y\right)dy+R. \tag{34}$$

From the expression of the remainder term $R$, from formula (27) and from the estimation (25) we have,
$$R\leq c\left\|\delta v\right\|_{L_2(D)}^2. \tag{35}$$

Then from formula (34) and estimation (35) we get that the functional (21) is Freshet differentiable and for the differential of the functional we have the expression

$$< J'_\beta (v) , \delta v >= \int\limits_\Omega [\psi (l, y) + \beta v (y)] \delta v (y) dy. \tag{36}$$

Show that the mapping $v \to J'_\beta (v)$ ,determined by this equality continuously acts from $V$ to the space $L_2 (D)$.

Let $\delta\psi (x, y) = \psi (x, y; v + \delta v) - \psi (x, y; v)$. If follows from (28)-(30) that $\delta\psi (x, y)$ is a gene-ralized solution from $W_2^1(\Omega)$ of the boundary value problem

$$\frac{\partial^2 \delta\psi}{\partial x^2} + \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i} \left( a_{ij} (y) \frac{\partial \delta\psi}{\partial y_j} \right) - a (y) \delta\psi = 0, \ \ (x, y) \in \Omega,$$

$$\frac{\partial \delta\psi(0, y)}{\partial x} = -\delta u (0, y) , \frac{\partial \delta\psi(l, y)}{\partial x} = 0, y \in D,$$

$$\left. \frac{\partial \delta\psi}{\partial \nu_A} \right|_{\partial D} = 0, x \in (0, l) .$$

For solution this problem as in (25) we get the estimation

$$\|\delta\psi\|_{W_2^1(\Omega)} \leq c \|\delta u (0, y)\|_{L_2(D)} . \tag{37}$$

Then from the last inequality of (25) and from inequality (37) we get the estimation

$$\|\delta\psi\|_{W_2^1(\Omega)} \leq c \|\delta v\|_{L_2(D)} . \tag{38}$$

By the imbedding theorem [6, p.149] we get the inequality

$$\|\delta\psi (l, y)\|_{L_2(D)} \leq c \|\delta\psi\|_{W_2^1(\Omega)} . \tag{39}$$

Therefore, inequalities (38), (39) yield the estimation

$$\|\delta\psi (l, y)\|_{L_2(D)} \leq c \|\delta v\|_{L_2(D)} . \tag{40}$$

Furthermore, using (36) and (40) it is easy to obtain the inequality

$$\left\|J'_\beta (v + \delta v) - J'_\beta (v)\right\|_{L_2(D)} \leq c \left[ \|\delta\psi (l, y)\|_{L_2(D)} + \|\delta v (y)\|_{L_2(D)} \right] \leq c \|\delta v\|_{L_2(D)} .$$

The right hand side of this inequality tends to zero as $\|\delta v\|_{L_2(D)} \to 0$. Hence it follows that $v \to J'_\beta(v)$ is a continuous mapping from $V$ to $L_2 (D)$.

Thus, we proved the following theorem.

**Theorem 4.** *Let the conditions imposed on the data of problem (7)-(9), (21) be fulfilled. Then the functional (21) is continuously Frechet differentiable on $V$ and its differential at the point $v \in V$ for the increment $\delta v \in L_2 (D)$ determined by the expression (36).*

**Theorem 5.** *Let the conditions of theorem 4 be fulfilled. Then for the optimality of the control $v_* \in V$ in problem (7)-(9), (21) it is necessary and sufficient that the inequality*

$$\int\limits_D [\psi_* (l, y) + \beta v_* (y)] (v (y) - v_* (y)) dy \geq 0 \tag{41}$$

*be fulfilled for any $v = v (y) \in V$, where $\psi_* (x, y)$ is the solution of problem (28)-(30) for $v = v_* (y)$.*

*Proof.* The set $V$ is convex in $L_2(D)$, the functional (21) strongly convex in $L_2(D)$ and by theorem 4 is continuously Frechet differentiable on $V$ and its differential at the point $v \in V$ is determined by the equality (36). Then by theorem 5 from Vasil'ev (1981), p. 28 on the element $v_*(y) \in V$ the fulfillment of the inequality $< J'_\beta(v_*), v - v_* >_{L_2(D)} \geq 0$ for all $v \in V$ is necessary and sufficient. Hence and from (36) if follows the validity of inequality (41) for all $v \in V$. Theorem 5 is proved. □

**Remark 1.** *Similar results are valid also for a problem when in the problem (1)- (3) instead of the condition (3) the condition* $u|_{\partial D} = 0, x \in (0,l)$ *and in the problem (7)-(9) instead of the conditions (8), (9) the conditions*

$$\frac{\partial u(0,y)}{\partial x} = 0, u(l,y) = v(y), y \in D,$$

$$u|_{\partial D} = 0, x \in (0,l).$$

*are taken.*

## 6 Conclusion

In this paper the problem of continuity for the second order linear elliptic equation is reduced to the optimal control problem, and the necessary and sufficient conditions of optimality are derived in the form of an integral inequality.

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